In this video, we will use linear modeling
to predict how a boat rocks from side
to side in the water around its flat equilibrium position.
In the previous video, we saw that we
could describe this rocking motion using the rotation
angle of the boat, theta, which is a function of time, t.
Recall that we defined theta to be
the angle between this flat waterline and this line
that rotates with the boat.

We use the convention that theta is positive
when the boat is rotated clockwise
and that theta is negative when the boat is
rotated counterclockwise.
We also saw that when the boat is sitting
flat in the water at theta equals 0,
it is in an equilibrium position.

So how do we determine the function theta of t?
Well, we know that the motion of the boat
is governed by Newton's second law.

Recall that for linear motion, Newton's second law
says that the mass, m, times acceleration, x double dot,
of an object is equal to the force, F,
applied to that object.
However, we are interested in the rotational motion
of the boat, and so we need to use the rotational form
of Newton's second law.

This states that the moment of inertia, I, times the angular
acceleration, theta double dot, of an object
is equal to the torque, tau, applied to that object.
Note that the moment of inertia is the rotational analog
of mass, and so the larger the moment of inertia of an object
is, the harder it will be to rotate.
Similarly, the torque is the rotational analog of force,
and so if you apply a larger torque to an object,
you will cause a greater angular acceleration.
We now have a second-order differential equation
whose solution is theta of t.
For a given boat, the moment of inertia will be a constant.
However, the torque will be a function of theta.
Therefore, before we can solve this differential equation,
we need to figure out how the torque depends on theta.
To do this, let's recall our observations
of the rocking boat in the previous video.
We saw that when the boat was sitting at theta equals 0,
in its equilibrium position, it just
sat there and did not rotate.
So the angular acceleration of the boat is 0.
And Newton's second law then tells us
that the torque when theta equals 0 is equal to 0.

So if I draw a graph of the torque versus theta,
this tells us that the torque must
pass through the origin of this graph.

We also observed that our boat had a stable equilibrium.

This was because when the boat was rocking in the water,
we observed that the water around the boat
acted like a spring.
For positive rotation angles, the boat
experienced a negative torque, acting
to rotate the boat back towards its equilibrium position.
So on this graph, it means that the torque
passes through the bottom right quadrant.
Similarly, we observe that for negative rotation angles,
the boat experienced a positive torque, again,
acting to rotate the boat back towards its equilibrium
position.
So we also know on this plot that the torque passes
through the top left quadrant.
In general, the torque will be a nonlinear function of theta.
And it might look like this, for example.

However, we are mostly interested in the boat's
behavior around its equilibrium of theta equals 0.
Therefore all we really care about
is how the torque behaves for small rotation angles
around this equilibrium.
Therefore we can approximate this nonlinear function using
a tangent line at theta equals 0.

This gives a linear approximation
for the torque of negative k times theta.
k is a positive constant such that negative k is
the slope of this tangent line.

Remember that this linear approximation
is only valid for small rotation angles.
We can now plug this linear approximation back
into our second-order differential equation
we obtained using Newton's second law.
This gives I theta double dot plus k theta equals 0.

You will recognize that this is the equation
of a simple harmonic oscillator.
And so we know that it yields oscillatory solutions
around theta equals 0.

We also know that the period, p, of these oscillations
is equal to 2 pi divided by the square root of k over I.
I should point out that if, in the future,
we wanted to perform a more detailed analysis
of the system, we should include a damping term
in this equation.

This is because we observed in the previous video
that when the boat was rocking, the amplitude of the boat's
oscillations decayed over time.
However, for this video, we will neglect the effects of damping
and just concentrate on this simple equation,
as it still describes the motion of the boat very well.
Up until this point, we have been considering a boat
that has a stable equilibrium.

However, this will not always be the case,
because we will later see that the shape of the boat
dictates whether the boat has a stable or unstable equilibrium.
In the case of an unstable equilibrium,
the torque near that equilibrium will have a positive slope
instead of a negative slope.
This positive slope will mean that this negative sign becomes
a positive sign, and so this positive sign becomes
a negative sign, giving the equation I
theta double dot minus k theta equals 0.

This differential equation yields very different
solutions.
For example, consider the initial conditions
theta at time t equals 0 equals 0 and theta dot at time t
equals 0 equals some constant c.
These initial conditions correspond
to the boat starting flat in the water
but with some initial angular velocity-- for example,
if you gave the boat a slight kick.
I now invite you to pause the video
and try and solve this differential equation
with these initial conditions.

Welcome back.
If you solve this differential equation
with these initial conditions, you
will find that theta t grows exponentially.

This means that if your boat has an unstable equilibrium,
as soon as you put it in the water,
it will start tipping over to one side.
Therefore, if you were designing the shape of a new boat,
it is incredibly important to ensure
that your boat has a stable equilibrium,
because if it has an unstable equilibrium, when it enters
the water, it will tip to one side
and would be at risk of capsizing.
Of course, once the boat tips far enough away
from equilibrium, we would need to use a different model
to predict the boat's behavior, since our model
is only valid for small rotation angles.
All that remains is to figure out how the constant k depends
on the shape of the boat.
This will allow us to analyze a boat of any shape

to predict whether it will have a stable or unstable equilibrium and, if it has a stable equilibrium, to predict the period of the boat's oscillations around this equilibrium.

在这个视频中，我们将使用线性建模来预测船只在水中平衡位置周围左右摇晃的情况。在之前的视频中，我们看到可以用船只的旋转角度θ来描述这种摇摆运动，θ是时间t的函数。回想一下，我们定义θ为平衡水线与随船旋转的线之间的角度。

我们采用的约定是，当船只顺时针旋转时，θ为正；当船只逆时针旋转时，θ为负。我们还看到，当船只以θ等于0的状态平放在水中时，它处于平衡位置。

那么，我们如何确定函数θ的t关系呢？嗯，我们知道船只的运动受到牛顿第二定律的控制。

回想一下，对于线性运动，牛顿第二定律说，物体的质量m乘以物体的加速度x双点，等于作用于物体的力F。然而，我们关心的是船只的旋转运动，因此我们需要使用牛顿第二定律的旋转形式。

它表明，物体的转动惯量I乘以物体的角加速度θ双点，等于作用于物体的力矩τ。请注意，转动惯量是质量的旋转类比，因此物体的转动惯量越大，旋转越困难。类似地，力矩是力的旋转类比，因此如果你对物体施加更大的力矩，你会导致更大的角加速度。现在我们有了一个二阶微分方程，其解为θ的t关系。对于给定的船只，转动惯量将是一个常数。然而，力矩将是θ的函数。因此，在我们解决这个微分方程之前，我们需要弄清楚力矩如何取决于θ。为了做到这一点，让我们回顾一下上一个视频中关于摇摆船只的观察结果。我们发现，当船只坐落在θ等于0的位置时，即平衡位置时，它只是静静地停在那里，没有旋转。因此，船只的角加速度为0。牛顿第二定律告诉我们，当θ等于0时，力矩等于0。因此，如果我画一个力矩与θ的图表，这就告诉我们力矩必须通过这个图表的原点。

我们还观察到，我们的船只具有稳定平衡状态。这是因为当船只在水中摇晃时，我们观察到船只周围的水起到了弹簧的作用。对于正旋转角度，船只经历了一个负力矩，作用于船只将其旋转回平衡位置。因此，在这个图表中，这意味着力矩通过右下象限。同样，我们观察到，对于负旋转角度，船只经历了一个正力矩，同样作用于船只将其旋转回平衡位置。所以我们也知道在这个图表中，力矩通过左上象限。通常，力矩将是θ的非线性函数。它可能是这样的，例如。

然而，我们对船只在θ等于0平衡位置附近的行为更感兴趣。因此，我们真正关心的是力矩在该平衡点附近的小旋转角度下的行为。因此，我们可以使用在θ等于0处的切线来近似这个非线性函数。

这给出了力矩的线性近似：负k乘以θ。k是一个正常数，使得负k是这个切线的斜率。

请记住，这个线性近似只对于小旋转角度有效。现在，我们可以将这个线性近似代入我们使用牛顿第二定律得到的二阶微分方程中。这给出了Iθ双点加kθ等于0。

你会认出这就是简谐振动的方程。因此，我们知道这个方程在θ等于0附近会产生周期性的解。

我们还知道这些振动的周期p等于2π除以k除以I的平方根。我应该指出，在未来，如果我们想对系统进行更详细的分析，我们应该在这个方程中加入阻尼项。这是因为我们观察到在前一个视频中，当船只摇晃时，船只振幅会随着时间的推移衰减。然而，在这个视频中，我们将忽略阻尼的影响，只专注于这个简单的方程，因为它仍然很好地描述了船只的运动。到目前为止，我们一直考虑的是一个具有稳定平衡的船只。

然而，情况并不总是如此，因为后面我们将看到船只的形状决定了船只是否具有稳定或不稳定的平衡。在不稳定平衡的情况下，该平衡点附近的力矩将有一个正斜率，而不是负斜率。这个正斜率意味着负号变为正号，所以这个正号变为负号，得到了方程Iθ双点减kθ等于0。

这个微分方程产生非常不同的解。例如，考虑初始条件θ在时间t等于0处等于0，θ点在时间t等于0处等于某个常量c。这些初始条件对应于船只开始时平放在水中，但具有一些初始角速度–例如，如果你给船只一个小推力。我现在邀请你暂停视频，并尝试用这些初始条件解这个微分方程。

欢迎回来。

如果你用这些初始条件解这个微分方程，你会发现θ随时间呈指数增长。

这意味着如果你的船只具有不稳定的平衡，一旦你把它放在水中，它就会开始向一侧倾斜。因此，如果你正在设计一个新船的形状，确保你的船只有稳定的平衡是非常重要的，因为如果它具有不稳定的平衡，在进入水中时它将向一侧倾斜，并有倾覆的风险。

当然，一旦船只离开平衡点足够远，我们就需要使用另一个模型来预测船只的行为，因为我们的模型只对小旋转角度有效。

剩下的工作就是弄清楚常数k如何取决于船只的形状。这将使我们能够分析任何形状的船只，预测它是否具有稳定或不稳定的平衡，并且如果它具有稳定的平衡，预测船只绕这个平衡点的振动周期。